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A MULTIPLE SCALE SOLUTION FOR CIRCULAR CYLINDRICAL SHELLS

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Abstract—A solution is presented for the differential equation governing the axisymmetric small deformation of a circular cylindrical shell having a thickness that varies in an arbitrary manner along its axis. The method of analysis used is Cochran's method of multiple scales which is a formalized technique for solving singular perturbation problems. The results are compared with known closed form solutions where such solutions exist.

NOTATION

The following symbols are used in this paper:

a	radial distance to the middle surface of the cylinder
$A, B, C, D, A_{0_1}, B_{0_1}, C_{0_1}, D_{0_1}$	constants of integration
ε	a small perturbation parameter
f	non-dimensionalized thickness function (t/t_0)
F, G, H	functions of f
g	a function of order $\varepsilon^{\frac{1}{2}}$ in the boundary layer
L	the length of the cylinder
t	the thickness of the cylinder on a transverse cross-section
t_0	a reference thickness of the cylinder
x	distance measured from the edge of the shell along its axis
$\bar{x}$	non-dimensionalized displacement (x/a)
ξ, η, ζ	independent variables

1. INTRODUCTION

A SINGULAR perturbation problem results from an attempt to apply a straightforward perturbation technique to the differential equation governing the axisymmetric linear deformation of a circular cylinder. This problem is solved herein by a technique due to Cochran [1] which is called the method of multiple scales [2]. The solution is for a circular cylinder having a thickness that varies in an arbitrary, but smooth, manner along the axis of the shell.

Asymptotic solutions exist for the equations governing the axisymmetric small deformation of shells using the Langer method and Blumenthal's technique [3–5]. Cochran's method of multiple scales has been applied to other fields of interest [6, 7] but no record can be found of its application to shell problems. The intent of the present paper is to demonstrate the utility of Cochran's method in the area of shell analysis.

2. THE GOVERNING DIFFERENTIAL EQUATION

The non-dimensionalized homogeneous differential equation governing the axisymmetric linear deformation of a circular cylinder having a thickness that varies in an arbitrary manner along its axis is [8]

$$\varepsilon^2[f^3(\bar{x})\omega''(\bar{x})]''+f(\bar{x})\omega(\bar{x}) = 0, \tag{1}$$

where

$$\varepsilon^2 = \frac{1}{12(1-\nu^2)}\frac{t_0^2}{a^2} \ll 1, \tag{2}$$

$$f = \frac{t}{t_0}, \tag{3}$$

$$(\,)' = \frac{d(\,)}{d\bar{x}}, \tag{4}$$

$$\bar{x} = \frac{x}{a}, \tag{5}$$

ω is the radial displacement of a point on the middle surface, t is the thickness of the shell on a particular transverse cross-section of the shell, x is the distance measured from the edge of the shell along its axis, t_0 is some reference thickness of the shell, ν is Poisson's ratio, and a is the radial distance to the middle surface of the shell.

Equation (1) can be put in a more convenient form by letting

$$f_1 = f^3, \tag{6a}$$

$$f_2 = 6f^2f', \tag{6b}$$

and

$$f_3 = 6(f')^2f+3f^2f''. \tag{6c}$$

Then equation (1) becomes

$$\varepsilon^2[f_1\omega''''+f_2\omega'''+f_3\omega'']+f\omega = 0. \tag{7}$$

It is assumed that the thickness function f is a well behaved function of order one in the region of interest.

3. SOLUTION BY THE MULTIPLE SCALE METHOD

An attempt to apply a straightforward perturbation technique to (1) fails due to a loss of higher derivatives, and a corresponding loss in boundary conditions, in the differential equation governing the first approximation. The region where the boundary conditions cannot be applied is frequently called a boundary layer.

For the first part of the analysis it is assumed that the boundary layer occurs at $\bar{x}$ equal to zero only. This is only done for convenience and is not a limitation of the method.

Following Cochran [1], this singular perturbation problem is solved by selecting the new set of independent variables:

$$\eta = \frac{g(\zeta)}{\varepsilon^{\frac{1}{2}}}, \tag{8}$$

and

$$\zeta = \bar{x}, \tag{9}$$

together with the asymptotic expansion

$$\omega(\bar{x}, \varepsilon) \sim \sum_{m=0}^{\infty} \omega_m(\zeta, \eta)\varepsilon^{m/2}. \tag{10}$$

The new independent variable η is of order one near $\bar{x}$ equal to zero and corresponds to the inner variable in the method of inner and outer expansions [2]. The variable ζ corresponds to the outer variable in the latter method.

Substituting (10) into equation (1) and equating to zero the coefficients of like powers of ε results in the following system of equations:

first order terms (ε^0):

$$(g')^4 f_1 \omega_{0\eta\eta\eta\eta} + f\omega_0 = 0, \tag{11a}$$

second order terms ($\varepsilon^{\frac{1}{2}}$):

$$(g')^4 f_1 \omega_{1\eta\eta\eta\eta} + f\omega_1 = -F_0 \omega_{0\eta\eta\eta} - F_1 \omega_{0\zeta\eta\eta\eta}, \tag{11b}$$

third order terms (ε):

$$\begin{aligned}(g')^4 f_1 \omega_{2\eta\eta\eta\eta} + f\omega_2 = & -F_0 \omega_{1\eta\eta\eta} \\ & - F_1 \omega_{1\zeta\eta\eta\eta} - F_2 \omega_{0\eta\eta} - F_3 \omega_{0\zeta\eta\eta} - F_4 \omega_{0\zeta\zeta\eta\eta},\end{aligned} \tag{11c}$$

where

$$F_0 = 6(g')^2 g'' f_1 + (g')^3 f_2, \tag{12a}$$

$$F_1 = 4(g')^3 f_1, \tag{12b}$$

$$F_2 = 4g'g''' f_1 + 3(g'')^2 f_1 + 3g'g'' f_2 + (g')^2 f_3, \tag{12c}$$

$$F_3 = 12g'g'' f_1 + 3(g')^2 f_2, \tag{12d}$$

and

$$F_4 = 6(g')^2 f_1. \tag{12e}$$

The subscripts η and ζ in the above equations denote partial differentiation with respect to the corresponding variable.

3.1 *First approximation*

The first approximation to the problem is obtained from the bounded solution of (11a) which is

$$\omega_0 = [C_0(\zeta)\cos\beta\eta + D_0(\zeta)\sin\beta\eta]e^{-\beta\eta} \tag{13}$$

where

$$\beta^4 = \frac{1}{4(g')^4 f^2} \tag{14}$$

the functions $g(\zeta)$, $C_0(\zeta)$ and $D_0(\zeta)$ are determined on the basis of what will be referred to as Lighthill's principle. Lighthill's principle requires that, "each approximation shall be no more singular than its predecessor—or vanish no more slowly—as $\varepsilon \to 0$ for arbitrary values of the independent variables" [2].

Applying Lighthill's principle to the second order problem (11b) gives:

$$0 = \omega_{1\eta\eta\eta\eta} + 4\beta^4\omega_1 \tag{15}$$

and

$$0 = [6(g')^2 g'' f_1 + (g')^3 f_2]\omega_{0\eta\eta\eta\eta} + 4(g')^3 f_1 \omega_{0\zeta\eta\eta}. \tag{16}$$

Equation (16) must hold for arbitrary η. Thus, substituting (13) into (16) and setting coefficients of $e^{-\beta\eta}\cos\beta\eta$, $\eta e^{-\beta\eta}\cos\beta\eta$, $e^{-\beta\eta}\sin\beta\eta$, and $\eta e^{-\beta\eta}\sin\beta\eta$ equal to zero gives:

$$0 = \beta', \tag{17a}$$

$$0 = GC_0 + C_0', \tag{17b}$$

$$0 = GD_0 + D_0', \tag{17c}$$

where

$$G = \frac{3}{4}\frac{f'}{f}. \tag{18}$$

These equations have been simplified by using the relations given in (6) and (14).

Equation (17a) establishes the condition that β must be a constant. Since it is an arbitrary constant, there is no loss in generality if it is set equal to one. Substituting β equal to one in (14) and integrating to find g, the form of η, as given by (8), then becomes

$$\eta = \frac{1}{(2\varepsilon)^{\frac{1}{2}}}\int_0^{\bar{x}} \frac{d\zeta}{f^{\frac{1}{4}}} \tag{19}$$

From (17b, c) it is found that

$$C_0 = \frac{C}{f^{\frac{3}{4}}}, \tag{20a}$$

and

$$D_0 = \frac{D}{f^{\frac{3}{4}}} \tag{20b}$$

where C and D are constants of integration to be determined from the boundary conditions at $\bar{x}$ equal to zero.

3.2 *Second approximation*

The solution of the second order problem, as represented by equation (15), is

$$\omega_1 = [C_1(\zeta)\cos\eta + D_1(\zeta)\sin\eta]e^{-\eta}. \tag{21}$$

It is then resolved, on the basis of applying Lighthill's principle to (11c), that

$$C_1 = [C_{01} - (C+D)F]\frac{1}{f^{\frac{3}{4}}} \tag{22a}$$

and

$$D_1 = [D_{01} + (D-C)F]\frac{1}{f^{\frac{3}{4}}} \tag{22b}$$

in which

$$F = F(\zeta) = \int \left[\frac{7}{16}\frac{f''}{f^{\frac{1}{2}}} + \frac{3}{64}\frac{(f')^2}{f^{\frac{3}{2}}}\right] \mathrm{d}\zeta. \tag{23}$$

C_{01} and D_{01} are constants of integration to be determined from the boundary conditions at $\bar{x}$ equal to zero.

3.3 *Summary of results*

So far the analysis has been based on the assumption that the boundary layer is at $\bar{x}$ equal to zero. If the boundary layer were at the edge furthest from $\bar{x}$ equal to zero, say x equal to L, the solution would be developed in the same fashion. These two solutions could then be superimposed to find the solution of the homogeneous differential equation (1) for a circular cylinder with a boundary layer at both ends.

Introducing the nondimensionalized coordinate

$$\xi = \frac{L-x}{a} \tag{24}$$

it is obvious that the two term asymptotic solution to the differential equation (1) for a boundary layer at ξ equal to zero and also $\bar{x}$ equal to zero is

$$\omega \sim \omega_0 + \omega_1 \varepsilon^{\frac{1}{2}} \tag{25}$$

where

$$\begin{aligned}\omega_m = {} & [A_m(\xi)\cos\eta(\xi) + B_m(\xi)\sin\eta(\xi)]e^{-\eta(\xi)} \\ & + [C_m(\bar{x})\cos\eta(\bar{x}) + D_m(\bar{x})\sin\eta(\bar{x})]e^{-\eta(\bar{x})},\end{aligned} \tag{26}$$

$$\eta(z) = \frac{1}{(2\varepsilon)^{\frac{1}{2}}}\int_0^z \frac{\mathrm{d}\zeta}{f^{\frac{1}{4}}(\zeta)}, \tag{27}$$

$$A_0(\xi) = AH, \qquad B_0(\xi) = BH, \tag{28a, b}$$

$$C_0(\bar{x}) = CH, \qquad D_0(\xi) = DH, \tag{28c, d}$$

$$A_1(\xi) = [A_{01} - (A+B)F(\xi)]H, \tag{28e}$$

$$B_1(\xi) = [B_{01} + (B-A)F(\xi)]H, \tag{28f}$$

$$C_1(\bar{x}) = [C_{01} - (C+D)F(\bar{x})]H, \tag{28g}$$

$$D_1(\bar{x}) = [D_{01} + (D-C)F(\bar{x})]H, \tag{28h}$$

and

$$H = H(z) = \frac{1}{f^{\frac{3}{4}}(z)}. \tag{29}$$

In the above, $A, B, C, D, A_{01}, B_{01}, C_{01}$ and D_{01} are constants of integration and z is set equal to ξ or $\bar{x}$ where appropriate. The function F is defined by (23) in terms of ζ which is replaced by ξ or $\bar{x}$ where necessary.

CONCLUSIONS

If the solution is a valid one it should reduce to or asymptotically approach, as $\varepsilon \to 0$, known closed form solutions of equation (1) for those cases where such closed form solutions exist. In particular, when f is a constant it is seen the solution is identically the known solution

$$\omega = [A \cos b\xi + B \sin b\xi]e^{-b\xi} + [C \cos b\bar{x} + D \sin b\bar{x}]e^{-b\bar{x}}$$

where

$$b = \frac{1}{(2\varepsilon)^{\frac{1}{2}}}.$$

The solution for f varying linearly in the form

$$f = 1 - \alpha\bar{x}$$

is in terms of Kelvin functions [9] and is

$$\omega = \frac{1}{f^{\frac{1}{2}}}[A\, ber' y + B\, bei' y]$$

for a boundary layer at $\bar{x}$ equal to zero where

$$y = \frac{2}{\alpha\varepsilon^{\frac{1}{2}}}\sqrt{(1-\bar{x}\alpha)},$$

$$\alpha = \left(\frac{t_0 - t_1}{t_0}\right)\frac{a}{L},$$

t_0 is the thickness at the edge x equal to zero and t_1 is the thickness at the edge x equal to L. If $\varepsilon \to 0$ the Kelvin functions can be written in the asymptotic series from [9] thus giving

$$\omega \sim \frac{1}{f^{\frac{3}{4}}}\left[C \cos \frac{y}{\sqrt{2}} + D \sin \frac{y}{\sqrt{2}}\right]e^{y/\sqrt{2}}$$

for the first term of the series. This is precisely the first approximation, ω_0, as obtained in the present solution.

To obtain the solution for other thickness variations, it is only necessary to substitute the thickness function, f, into equations (27), (28) and (29). Higher order approximations (e.g. $\omega_2, \omega_3, \ldots$) could be obtained by a continuation of the method, but for most problems only the first approximation, ω_0, is necessary.

The solution points out some of the unique characteristics of Cochran's method of multiple scales. In solving this problem it was not necessary to assume the form of either the solution or the inner variable as is commonly done in other methods. Although the method was used on an ordinary differential equation in this presentation, it has also been used in solving certain partial and nonlinear differential equations [1].

Work is presently being carried out to solve the complete axisymmetric equations for small deformations of orthotropic shells by Cochran's method.

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Абстракт—Дается решение для дифференциального уравнения, касающегося осесимметрической малой деформации, круглой цилиндрической оболочки. Толщина оболочки изменяется поризвольно вдоль ее оси. Для расчета применяется метод Кохрана многократных масштабов, который является определенным способом расчета одинарных задач пертурбации. Результаты сравниваются с известными решениями, в замкнутом виде, для случаев, где такие решения существуют.